[arXiv:2303.05532]
(to appear in Phys. Rev. Lett.)

Exceptional points are useless for quantum sensing

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"Workshop on Non-Equilibrium Dynamics", 14.11.2023, Nottingham

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FNP Fundacja na rzecz Nauki Polskiej
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Enhancing quantum sensors by exploiting the non-Hermitian description of their dynamics

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Operating quantum systems at singular (unstable) points within linear response theory: new prospects for sensing

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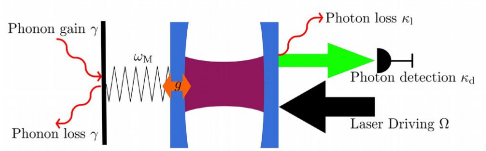
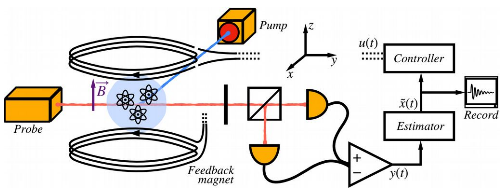
“Non-Hermitian” quantum sensors: motivation

Quantum-enhanced/entanglement-driven quantum sensors:



[*L.Clark*, *B.Markowicz* & *JK*, Quantum **6**, 812 (2022)]

AIM: Use *quantum effects* to boost the performance of a quantum sensor.



[J. Amoros-Binefa & *JK*, NJP **23** 123030 (2021)]

In particular, ***entanglement***:

separable states

$$\Delta^2 \tilde{B} \sim \frac{1}{N}$$

SQL

entangled states

$$\Delta^2 \tilde{B} \sim \frac{1}{N^2}$$

HL

PROBLEM: ***impact of noise.***

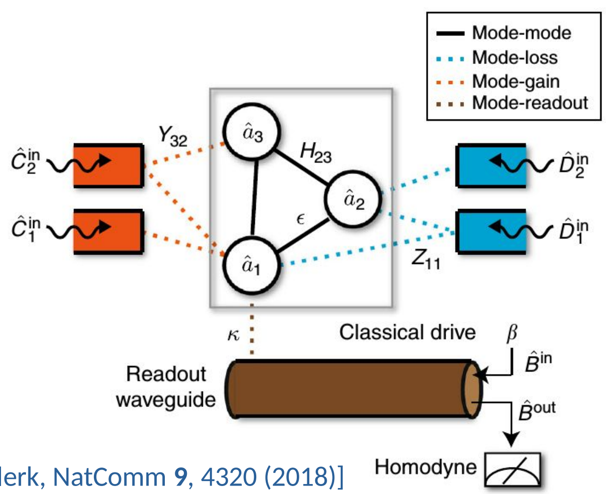
any local noise

$$\Delta^2 \tilde{B} \sim \frac{\mathrm{const}}{N}$$

CHANGE OF PARADIGM: “instability-tuned” dissipative quantum sensors:

AIM: *Control* and *fine-tune the noise*, so that the quantum system is *very sensitive to small perturbations*.

Linear *quantum sensors* within the **input-output** (*Langevin*) formalism:



[H.-K. Lau & A.A. Clerk, NatComm **9**, 4320 (2018)]

$$\hat{\boldsymbol{a}} := \{\hat{a}_1, \hat{a}_2, \dots\}^{\mathrm{T}}$$

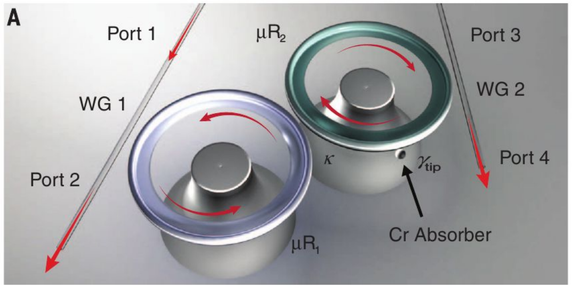
$$\partial_t \hat{\boldsymbol{a}} = -\mathrm{i}\,\mathbf{H}\,\hat{\boldsymbol{a}} + \hat{\boldsymbol{C}}_{\rm in} + \hat{\boldsymbol{D}}_{\rm in} + \hat{\boldsymbol{B}}_{\rm in},$$

Non-Hermitian dynamical generator

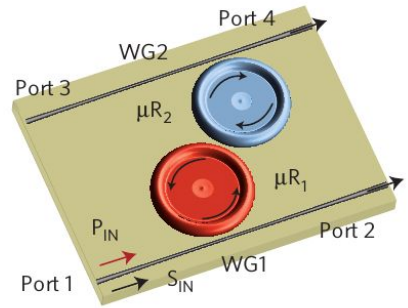
Measured **signal modes**:

$$\hat{B}_{\ell,\mathrm{out}}(t) = \hat{B}_{\ell,\mathrm{in}}(t) - \sqrt{\kappa}\,\hat{a}_\ell(t)$$

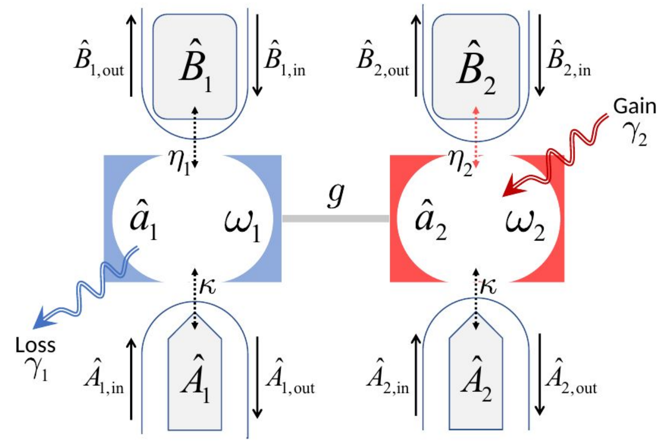
Canonical two-cavity system



[Peng et al, Science **346**, 328 (2014)]



[Peng et al, NatPhys **10**, 394 (2014)]



Reduced dynamics of the cavities with equal internal frequencies: $\omega_0 := \omega_1 = \omega_2$

$$\hat{H} = \omega_0(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + g(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \rho] + \gamma_1 \left(\hat{a}_1 \rho \hat{a}_1^\dagger - \frac{1}{2}\left\{\hat{a}_1^\dagger \hat{a}_1, \rho\right\}\right) + \gamma_2 \left(\hat{a}_2^\dagger \rho \hat{a}_2 - \frac{1}{2}\left\{\hat{a}_2 \hat{a}_2^\dagger, \rho\right\}\right)$$

⇕

$$\hat{\boldsymbol{a}} := \{\hat{a}_1, \hat{a}_2\}^{\mathrm{T}}$$

$$\partial_t \hat{\boldsymbol{a}} = -\mathrm{i}\left(\omega_0 \mathbf{I} + \mathbf{H}\right)\hat{\boldsymbol{a}}$$

$$\mathbf{H} = \begin{pmatrix} -\mathrm{i}\gamma_1 & g \\ g & +\mathrm{i}\gamma_2 \end{pmatrix}$$

Non-Hermitian dynamical generator

$$\lambda_\pm = -\mathrm{i}\gamma_- \pm \sqrt{g^2 - \gamma_+^2}, \quad |e_\pm) = \begin{pmatrix} -\mathrm{i}\gamma_+ \pm \sqrt{g^2 - \gamma_+^2} \\ g \end{pmatrix}, \quad \gamma_\pm = \frac{\gamma_1 \pm \gamma_2}{2}$$

Exceptional point (EP)
when both eigenvalues and eigenvectors coalesce $(g = \gamma_+)$

Consider perturbation @ an **EP**, e.g. internal frequency or loss rate:

$$\mathbf{H}' = \mathbf{H} + \epsilon \mathbf{V}$$

$$\mathbf{V} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

⇒ $$\Delta\lambda \sim \sqrt{\epsilon} + O(\epsilon^{3/2})$$

$$\Delta\lambda \sim \epsilon^{1/n}$$

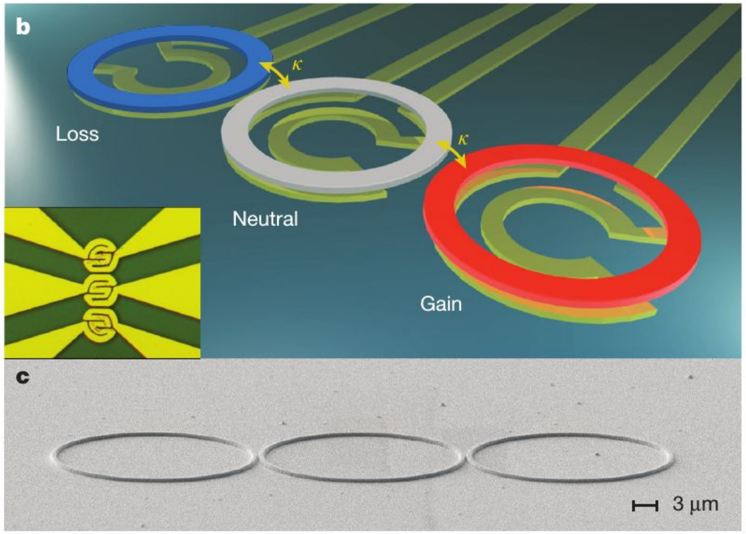
At an EP of the n^th-order

Infinitely steep signal in the limit ε→0 !!!

“EP-enhanced” sensing

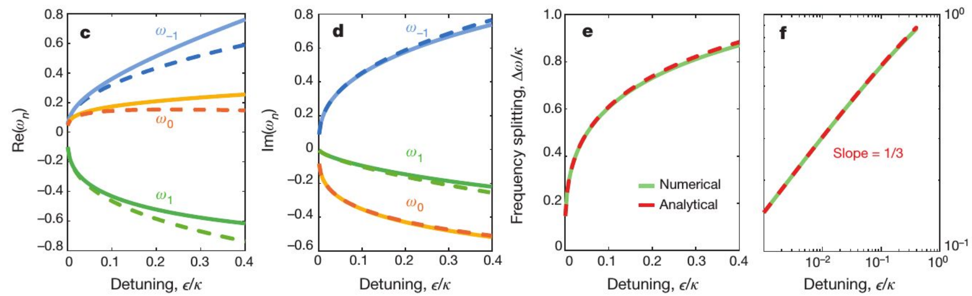
[Hodaei et al, Nature **548**, 187 (2017)]

Three-cavity system → 3rd order EP:

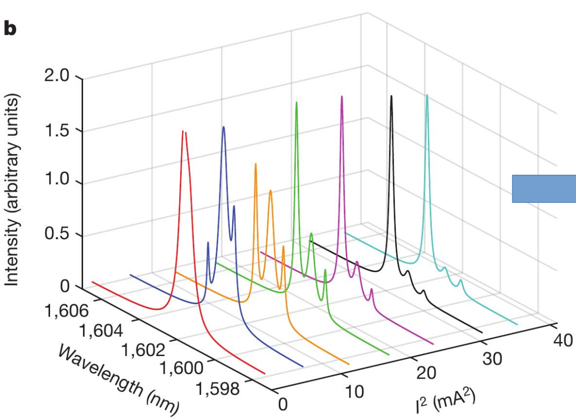


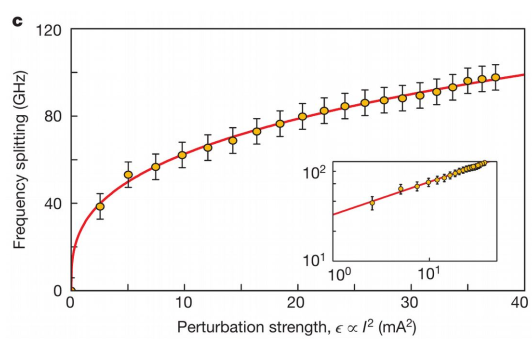
Perturbation of gain in the 1st cavity:

$$H = \begin{pmatrix} ig+\epsilon & \kappa & 0 \\ \kappa & 0 & \kappa \\ 0 & \kappa & -ig \end{pmatrix}$$



Spectral density measurement as a function of perturbation ($\varepsilon = I^2$):





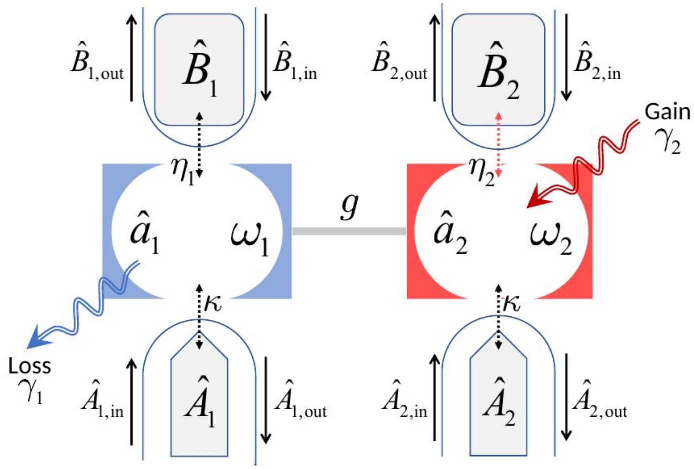
$$\Delta\lambda \sim \epsilon^{1/3}$$

Ok, signal is very sensitive (*steep slope*) **to small ε variations….**

But what is the level of (quantum) noise ???

Canonical two-cavity system: quantum model

[arXiv:2303.05532]



Interaction Hamiltonians

$$\hat{H}_{\hat{a}_1\hat{B}_1} = \sqrt{\eta_1}\,\hat{a}_1\hat{B}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_2\hat{B}_2} = \sqrt{\eta_2}\,\hat{a}_2^\dagger\hat{B}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_1\hat{A}_1} = \sqrt{\kappa}\,\hat{a}_1\hat{A}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_2\hat{A}_2} = \sqrt{\kappa}\,\hat{a}_2\hat{A}_2^\dagger + h.c.$$

Input-output (linear) dynamics including *probing (A)* and *scattering (B)* modes:

$$\hat{\boldsymbol{a}} := \{\hat{a}_1, \hat{a}_2\}^{\mathrm{T}} \qquad (\omega_0 := \omega_1 = \omega_2)$$

$$\partial_t \hat{\boldsymbol{a}} = -\mathrm{i}\left(\omega_0 \mathbf{I} + \mathbf{H}\right)\hat{\boldsymbol{a}} + \hat{\boldsymbol{A}}_{\mathrm{in}} + \hat{\boldsymbol{B}}_{\mathrm{in}}$$

As before: $\mathbf{H} = \begin{pmatrix} -\mathrm{i}\gamma_1 & g \\ g & +\mathrm{i}\gamma_2 \end{pmatrix}$

$$\gamma_1 = \frac{\eta_1 + \kappa}{2}, \quad \gamma_2 = \frac{\eta_2 - \kappa}{2}$$

Gaussian state of the output mode *measured at the frequency* ω:

$$\hat{A}_{\ell,\mathrm{out}}[\omega] := \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \hat{A}_{\ell,\mathrm{out}}(t)$$

"***on resonance***" measurement: $(\omega = \omega_0)$

Gaussian input-output relations for mode at frequency ω:

mean: $\mathsf{S}^A_{\mathrm{out}} = (\mathsf{I} - \kappa \mathsf{G})\,\mathsf{S}^A_{\mathrm{in}}$

variance: $\mathsf{V}^A_{\mathrm{out}} = (\mathsf{I} - \kappa \mathsf{G})\,\mathsf{V}^A_{\mathrm{in}}\,(\mathsf{I} - \kappa \mathsf{G})^{\mathrm{T}} + \kappa \mathsf{G}\Xi\tilde{\mathsf{V}}^B_{\mathrm{in}}\Xi^{\mathrm{T}}\mathsf{G}^{\mathrm{T}}$

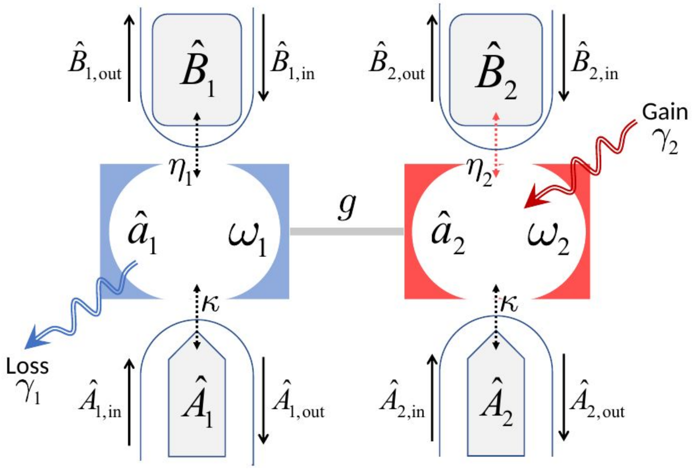
Central object: ***linear response***

$$\mathsf{G}[\omega = \omega_0] = \mathsf{J}\left(\epsilon \mathsf{V} - \mathsf{H}\right)^{-1}$$

$$\mathsf{H} := \begin{pmatrix} \Re[\mathbf{H}] & -\Im[\mathbf{H}] \\ \Im[\mathbf{H}] & \Re[\mathbf{H}] \end{pmatrix}$$

Canonical two-cavity system: quantum model

[arXiv:2303.05532]



Interaction Hamiltonians

$$\hat{H}_{\hat{a}_1\hat{B}_1} = \sqrt{\eta_1}\,\hat{a}_1\hat{B}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_2\hat{B}_2} = \sqrt{\eta_2}\,\hat{a}_2^\dagger\hat{B}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_1\hat{A}_1} = \sqrt{\kappa}\,\hat{a}_1\hat{A}_1^\dagger + h.c.$$
$$\hat{H}_{\hat{a}_2\hat{A}_2} = \sqrt{\kappa}\,\hat{a}_2\hat{A}_2^\dagger + h.c.$$

Gaussian input-output relations:
(*mean, variance*)

$$\mathsf{S}^A_{\mathrm{out}} = (\mathsf{I} - \kappa\mathsf{G})\,\mathsf{S}^A_{\mathrm{in}}$$
$$\mathsf{V}^A_{\mathrm{out}} = (\mathsf{I} - \kappa\mathsf{G})\,\mathsf{V}^A_{\mathrm{in}}\,(\mathsf{I} - \kappa\mathsf{G})^{\mathrm{T}} + \kappa\mathsf{G}\Xi\tilde{\mathsf{V}}^B_{\mathrm{in}}\Xi^{\mathrm{T}}\mathsf{G}^{\mathrm{T}}$$

Gaussian formalism for sensing ***multiparameter*** linear perturbations $\boldsymbol{\theta} := \{\theta_0, \theta_1, \theta_2, \dots\}$

$$\mathsf{G} = \mathsf{J}\left(\epsilon\mathsf{V} - \mathsf{H}\right)^{-1} \quad \Rightarrow \quad \mathsf{G}_{\boldsymbol{\theta}} = \mathsf{J}\left(\sum_{i=0}^{m}\theta_i\mathsf{V}_i - \mathsf{H}\right)^{-1} = \mathsf{J}\left(\theta_0\mathsf{n}_0 - \mathsf{H}_{\bar{\boldsymbol{\theta}}}\right)^{-1}$$

nuisance

Quantum Cramer-Rao Bound for Gaussian states:

$$\boldsymbol{\Delta}^2\tilde{\boldsymbol{\theta}} := \mathbb{E}\left[(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathrm{T}}\right]$$

$$\nu\,\boldsymbol{\Delta}^2\tilde{\boldsymbol{\theta}} \geq \boldsymbol{F}^{-1} \geq \boldsymbol{\mathcal{F}}^{-1}$$

Fisher Information matrices:

classical (CFIM):

$$\boldsymbol{F}_{jk} = \frac{1}{2}\left[\mathsf{C}^{-1}(\partial_j\mathsf{C})\mathsf{C}^{-1}(\partial_k\mathsf{C})\right] + (\partial_j\bar{\mathsf{x}})^{\mathrm{T}}\mathsf{C}^{-1}(\partial_k\bar{\mathsf{x}})$$

e.g. heterodyne

$$\bar{\mathsf{x}} = \mathsf{S}^A_{\mathrm{out}}$$
$$\mathsf{C} = \mathsf{V}^A_{\mathrm{out}} + \mathsf{I}$$

"noisy" **quantum** (QFIM):

$$\boldsymbol{\mathcal{F}}_{jk} \approx \frac{1}{2}\left[(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_j\mathsf{V}^A_{\mathrm{out}})(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_k\mathsf{V}^A_{\mathrm{out}})\right] + (\partial_j\mathsf{S}^A_{\mathrm{out}})^{\mathrm{T}}(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_k\mathsf{S}^A_{\mathrm{out}})$$

Canonical two-cavity system: quantum model

[arXiv:2303.05532]

Linear response determines the behaviour of the **single-parameter QFI**:

$$\mathsf{G}_{\boldsymbol{\theta}} = \mathsf{J}\,(\theta_0 \mathsf{V}_0 - \mathsf{H})^{-1} \quad \Rightarrow \quad \mathcal{F}_{00} \approx \frac{1}{2}\left[(\mathsf{V}^A_{\rm out})^{-1}(\partial_0 \mathsf{V}^A_{\rm out})(\mathsf{V}^A_{\rm out})^{-1}(\partial_0 \mathsf{V}^A_{\rm out})\right] + (\partial_0 \mathsf{S}^A_{\rm out})^{\rm T}(\mathsf{V}^A_{\rm out})^{-1}(\partial_0 \mathsf{S}^A_{\rm out})$$

Result 1:

non-singular dynamical generator

$$\det \mathbf{H} \neq 0$$

linear response admits *Neumann series*:

$$\mathsf{G}_{\theta_0} = -\mathsf{J}\mathsf{H}^{-1}\Big(\mathsf{I} + \sum_{k=1}^{\infty} \theta_0^k (\mathsf{n}_0 \mathsf{H}^{-1})^k\Big)$$

$$\lim_{\theta_0 \to 0} \mathsf{G}_{\theta_0} = -\mathsf{J}\mathsf{H}^{-1}$$

no possibility for the divergence of the QFI i.e. *precision*.

$$\mathcal{F}_{00} \approx \mathrm{const} + O(\theta_0)$$

Consequence: Satisfying **EP** cannot have impact on its own. $(g = \gamma_+)$

[Chen et al, NJP **21**, 083002 (2019)]

$$(g^2 \neq \gamma_1 \gamma_2)$$

$$\mathbf{H} = \begin{pmatrix} -\mathrm{i}\gamma_1 & g \\ g & +\mathrm{i}\gamma_2 \end{pmatrix}$$

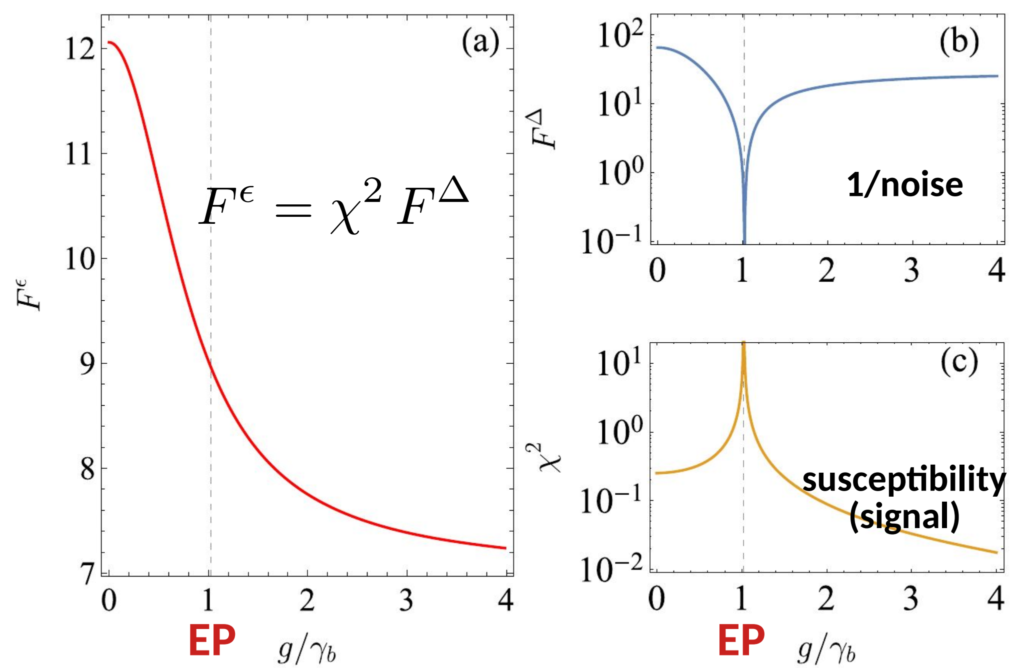
Gaussian input-output relations:
(mean, variance)

$$\mathsf{S}^A_{\rm out} = (\mathsf{I} - \kappa \mathsf{G})\,\mathsf{S}^A_{\rm in}$$

$$\mathsf{V}^A_{\rm out} = (\mathsf{I} - \kappa \mathsf{G})\,\mathsf{V}^A_{\rm in}\,(\mathsf{I} - \kappa \mathsf{G})^{\rm T} + \kappa \mathsf{G}\,\mathsf{V}^B_{\rm in}\,\kappa \mathsf{G}^{\rm T}$$

Take home lesson:
verify whether conclusions are *not* consequences of just the *mathematical description* itself, rather than the actual *physics of the system*.

Example: coordinates (Schwarzschild→Lemaître) in black-hole description.



Canonical two-cavity system: quantum model

[arXiv:2303.05532]

Linear response determines the behaviour of the **single-parameter QFI**:

$$\mathsf{G}_{\boldsymbol{\theta}} = \mathsf{J}\left(\theta_0 \mathsf{V}_0 - \mathsf{H}\right)^{-1} \;\Rightarrow\; \mathcal{F}_{00} \approx \frac{1}{2}\left[(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_0 \mathsf{V}^A_{\mathrm{out}})(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_0 \mathsf{V}^A_{\mathrm{out}})\right] + (\partial_0 \mathsf{S}^A_{\mathrm{out}})^{\mathrm{T}}(\mathsf{V}^A_{\mathrm{out}})^{-1}(\partial_0 \mathsf{S}^A_{\mathrm{out}})$$

Result 2:

singular dynamical generator

$$\det \mathbf{H} = 0$$

$$(g^2 = \gamma_1\gamma_2)$$

$$\mathbf{H} = \begin{pmatrix} -\mathrm{i}\gamma_1 & g \\ g & +\mathrm{i}\gamma_2 \end{pmatrix}$$

linear response must be expanded around the ***singularity*** (*Sain-Massey expansion*):

$$\mathsf{G}_{\theta_0} = \mathsf{J}\theta_0^{-s}\sum_{k=0}^{r}\theta_0^k \mathsf{X}_k \;\Rightarrow\; \mathcal{F}^{\mathrm{S}}_{00} \approx \theta_0^{-2s}\left(\mathrm{const} + O(\theta_0)\right)$$

QFI diverges at the rate given by the ***pole order*** *s* of the linear-response function

Cramer-Rao Bound:

$$\delta_{\mathrm{Q}}\theta_0 = \frac{1}{\sqrt{\mathcal{F}_{00}}} \propto \theta_0^s$$

Error is vanishing as $\theta_0 \to 0$,
i.e. perfect precision at a singularity

@ ***singularity:***

$$\mathbf{H}' = \mathbf{H} + \theta_0 \mathbf{V}$$

two-mode perturbation:

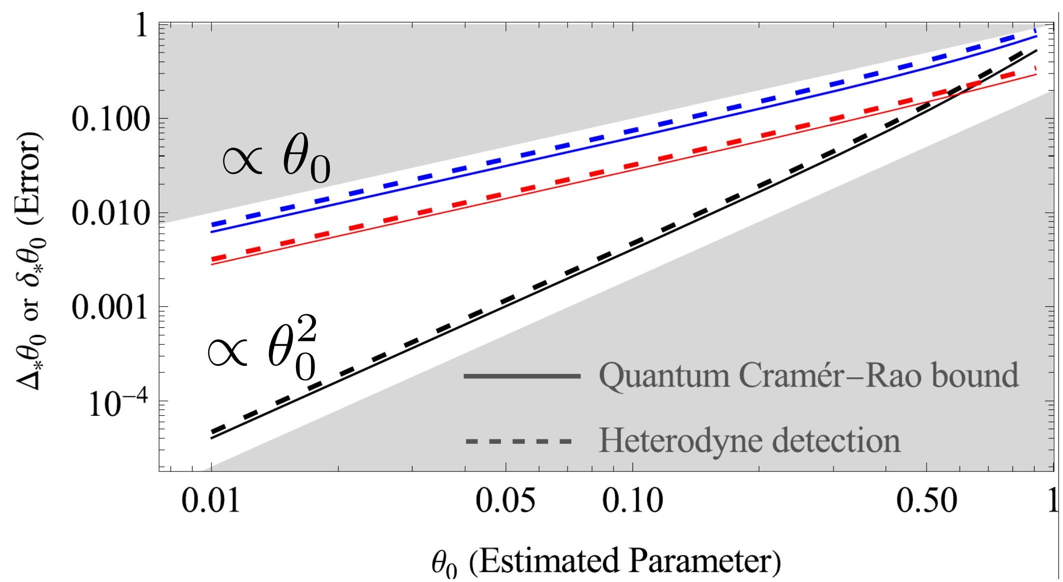
$$\mathbf{V} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\Rightarrow\; s = 2$$

pole order

single-mode perturbation:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \;\Rightarrow\; s = 1$$

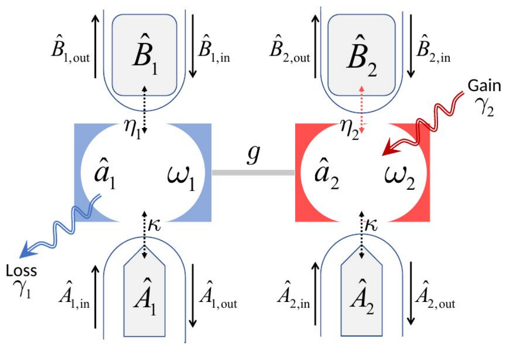
pole order



Canonical two-cavity system: quantum model

[arXiv:2303.05532]

Physical interpretation of the ***singularity-enhanced sensing***:



$$\hat{\boldsymbol{a}} := \{\hat{a}_1, \hat{a}_2\}^{\mathrm{T}} \qquad (\omega_0 := \omega_1 = \omega_2)$$

$$\partial_t \hat{\boldsymbol{a}} = -\mathrm{i}\left(\omega_0 \mathbf{I} + \mathbf{H}\right)\hat{\boldsymbol{a}} + \hat{\boldsymbol{A}}_{\mathrm{in}} + \hat{\boldsymbol{B}}_{\mathrm{in}}$$

$$\mathbf{H} = \begin{pmatrix} -\mathrm{i}\gamma_1 & g \\ g & +\mathrm{i}\gamma_2 \end{pmatrix} \qquad \lambda_\pm = -\mathrm{i}\gamma_- \pm \sqrt{g^2 - \gamma_+^2}$$

"***on resonance***" measurement: $(\omega = \omega_0)$

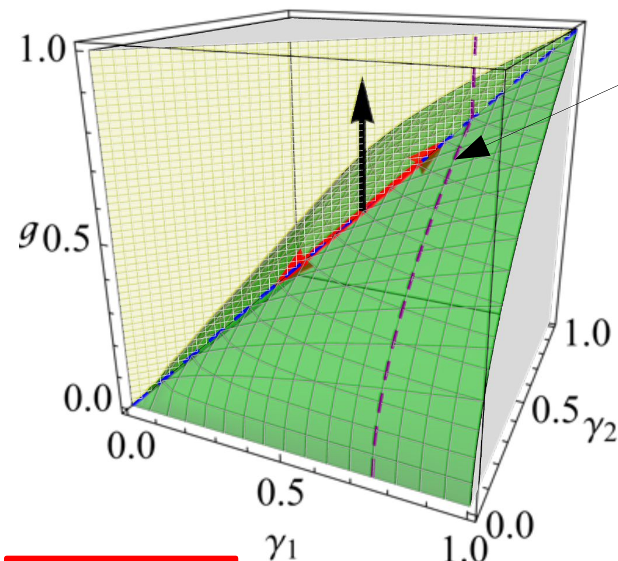
Different dynamical conditions in the space of parameters:

1) **Singularity** yields the *green surface* in the parameter space.
$$\det \mathbf{H} = 0 \quad (g^2 = \gamma_1\gamma_2)$$

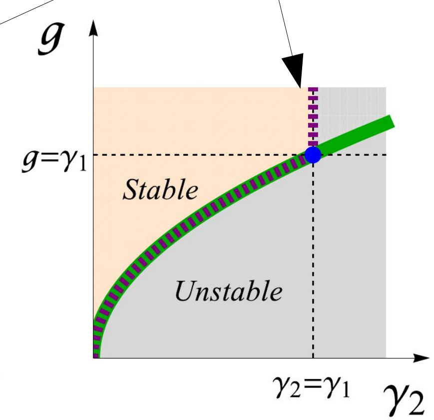
2) **Exceptional point** is nothing special, but works in the special case of a *balanced system* (*blue dashed line*). $(g = \gamma \text{ with } \gamma := \gamma_1 = \gamma_2)$

3) **IMPORTANT:** The system becomes unstable when eigenvalues have positive imaginary part (*grey regimes*), corresponding to the **lasing threshold**, but singularity is *not* equivalent.

4) If one introduces a **nuisance parameter**—whose variations (***red arrow***, in contrast to ***black arrow***) preserve the singularity, then the error behaviour gets degraded: $\delta_{\mathrm{Q}}\theta_0 \propto \theta_0^2 \Rightarrow \delta_{\mathrm{Q}}\theta_0 \propto \theta_0$



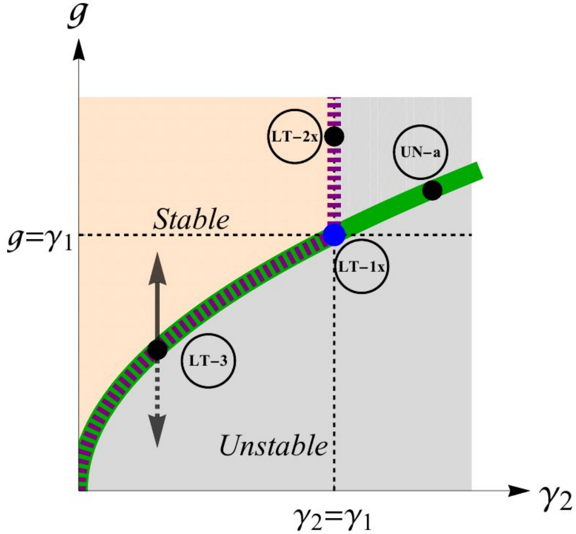
lasing threshold



Canonical two-cavity system: quantum model

[arXiv:2303.05532]

Different ways to ***operate*** (on a <u>singular point</u>!) *the system* v.s. ***perturb*** *the system*:



Perturbing the coupling constant g at the **lasing threshold** (LT):

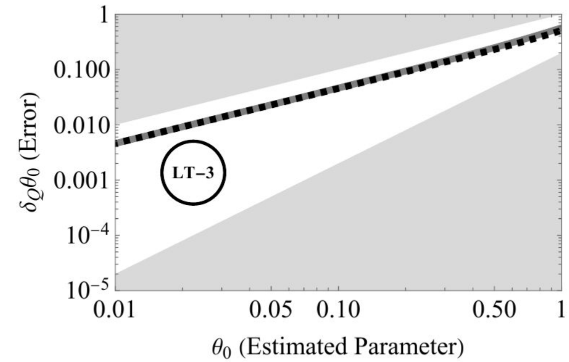
$$\mathsf{G}[\omega=\omega_0]=\mathsf{J}\,(\epsilon\mathsf{V}-\mathsf{H})^{-1}$$

$$\mathbf{H}=\begin{pmatrix}-i\gamma_1 & g\\ g & +i\gamma_2\end{pmatrix}$$

$$\mathbf{V}=\pm\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow s=1$$

pole order

$$\delta_Q\theta_0 \propto \theta_0$$



Generalise to measurements "*beyond resonance*"

$(\omega \neq \omega_0)$

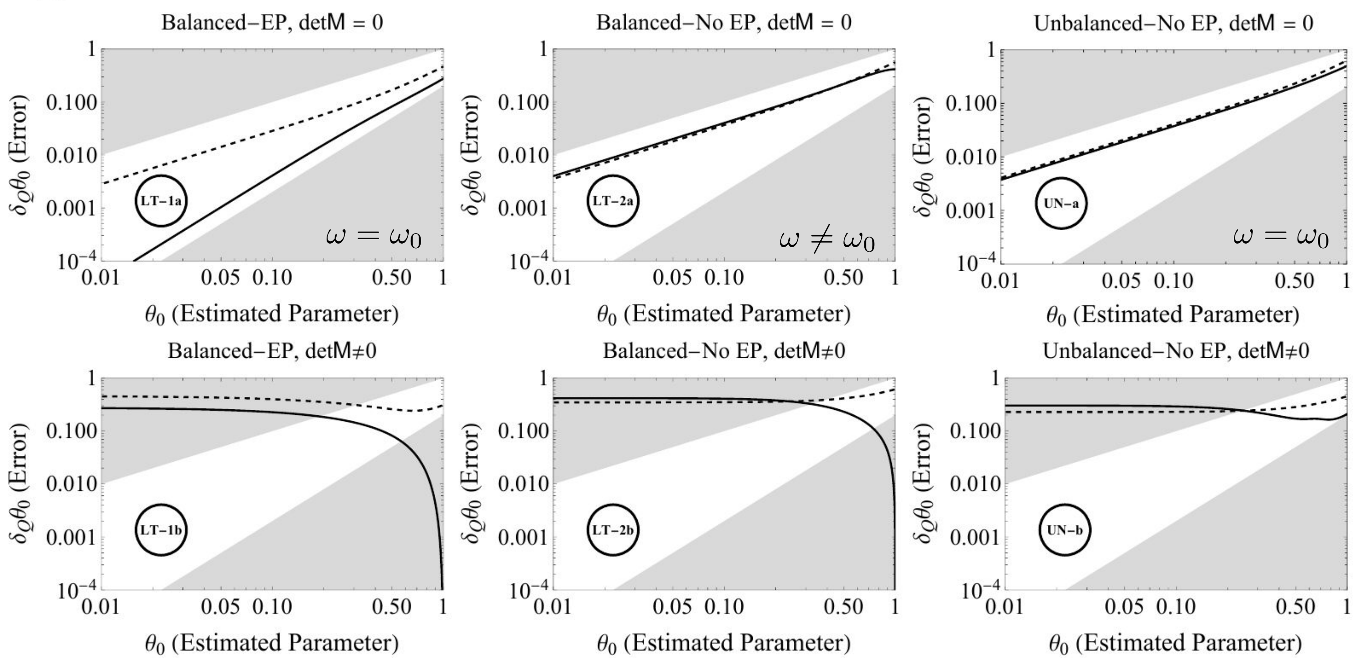
$$\mathsf{G}[\omega]=\mathsf{J}\,(\epsilon\mathsf{V}+\mathsf{M}[\omega])^{-1}$$

$$\mathsf{M}[\omega]=(\omega-\omega_0)\mathsf{I}-\mathsf{H}$$

now, the ***effective dynamical generator*** (in the Fourier space), whose <u>singularity is essential</u>!

two-mode freq. perturbation $\Rightarrow$ $\mathbf{V}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

single-mode freq. perturbation $\Rightarrow$ $\mathbf{V}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$



[arXiv:2303.05532]
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Summary

1) **Non-Hermitian sensors** involve systems described by **Hermitian Hamiltonians**!
...but the *nomenclature of non-Hermitian Hamiltonians* is used to describe dynamical generators...

2) It is *not* enough to study the behaviour of the measured signal:
i.e. it is always the **signal-to-noise ratio** (SNR) that correctly quantifies the precision.
Such an issue is naturally taken care of by classical/quantum **Fisher information** (FI).

3) It is the **singularity of the linear-response** (Green's) **function** that leads to the divergence of the FI – infinite precision at the singular point (*criticality*).

4) However, all this relies on **fine-tuning of the system**:
Even within the *local (frequentist) approach* to estimation the sensing capabilities depend on the ability to estimate other (irrelevant) parameters – **nuisance parameters**.

Thank you very much for your attention!

[arXiv:2303.05532]

Sain-Massey procedure: singular-matrix perturbations

Let $\mathsf{A}(z) = \mathsf{A}_0 + z\mathsf{A}_1 + z^2\mathsf{A}_2 + \cdots$ be an analytic matrix-valued function of z in some non-empty neighbourhood of $z = 0$, such that $\mathsf{A}^{-1}(z)$ exists in some (possibly punctured) disc centered at $z = 0$.

Then, $\mathsf{A}^{-1}(z)$ possesses a Laurent series expansion

$$\mathsf{A}^{-1}(z) = \frac{1}{z^s}\left(\mathsf{X}_0 + z\mathsf{X}_1 + z^2\mathsf{X}_2 + \dots\right),$$

where $\mathsf{X}_0 \neq 0$ and s is a natural number, known as the order of the pole at $z = 0$. The pole order s and the coefficient-matrices X_i can be determined by the *Sain-Massey* (SM) procedure as follows.

Firstly, compute the so-called *augmented matrix* as

$$\boldsymbol{\mathcal{A}}^{(t)} := \begin{pmatrix} \mathsf{A}_0 & 0 & 0 & \dots & 0 \\ \mathsf{A}_1 & \mathsf{A}_0 & 0 & \dots & 0 \\ \mathsf{A}_2 & \mathsf{A}_1 & \mathsf{A}_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathsf{A}_t & \mathsf{A}_{t-1} & \mathsf{A}_{t-2} & \dots & \mathsf{A}_0 \end{pmatrix},$$

and stop at t such that $\boldsymbol{\mathcal{A}}^{(t)} - \boldsymbol{\mathcal{A}}^{(t-1)} = n$, where n is the dimension of $\mathsf{A}(z)$. The *order of the pole* s is then given by such smallest possible value of t.

Secondly, compute the (Moore–Penrose) *pseudoinverse* of the augmented matrix at the pole level ($t = s$), i.e.:

$$\mathcal{G}^{(s)} := [\boldsymbol{\mathcal{A}}^{(s)}]^+.$$

Now, the first (coefficient) matrix in the expansion is given by the top-right block of the matrix $\mathcal{G}^{(s)}$, i.e. $\mathsf{X}_0 = \mathcal{G}^{(s)}_{0s}$ with

$$\mathcal{G}^{(s)} = \begin{pmatrix} \mathcal{G}^{(s)}_{00} & \dots & \mathcal{G}^{(s)}_{0s} \\ \vdots & \dots & \vdots \\ \mathcal{G}^{(s)}_{s0} & \dots & \mathcal{G}^{(s)}_{ss} \end{pmatrix}.$$

The higher X_k with $k = 1, 2, \dots$ are obtained with help of other blocks via the recursive formula:

$$\mathsf{X}_k = \sum_{j=0}^{s} \mathcal{G}^{(s)}_{0j}\left(\delta_{j+k,s}\mathsf{I} - \sum_{i=1}^{k} \mathsf{A}_{i+j}\mathsf{X}_{k-i}\right).$$